Maximal Rank-One Spaces of Matrices Over Chain Semirings. II. (u, *i*)-Spaces

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ABSTRACT

The work begun in Paper I is continued here. Two more classes of maximal rank-1 spaces of matrices are identified. One is another class of spaces formed by extending full factor spaces. The other, which applies to Boolean (0, 1) matrices only, is a class of maximal rank-1 spaces which contain no full factor spaces as subspaces.

1. INTRODUCTION

A *chain semiring* $\mathbb{S}$ is a linearly ordered set with minimum and maximum elements (denoted 0 and 1 respectively, $0 \neq 1$) and two operations defined as follows: $a + b = \max\{a, b\}$ and $ab = \min\{a, b\}$. The set of $m \times n$ matrices over $\mathbb{S}$ is denoted by $\mathscr{M}_{mn}(\mathbb{S})$. See Paper I [1] for other basic definitions.

In Paper I, we saw a method for extending the full left factor space $\mathfrak{V}_{\mathbf{u}}$ [1, Definition 3.7] generated by $\mathbf{u}$ to the maximal rank-1 space [1, Definition 3.2] $\mathfrak{W}_{\mathbf{u}}$. We called this extension space the $\mathbf{u}$-space [1, Definition 3.6] in $\mathscr{M}_{mn}(\mathbb{S})$. We observed that when $\max u_i = 1$, $\mathfrak{W}_{\mathbf{u}}$ is the only maximal rank-1 space containing $\mathfrak{V}_{\mathbf{u}}$. But if $\max u_i < 1$, we saw that $\mathfrak{V}_{\mathbf{u}} \subseteq \mathfrak{W}_{\mathbf{v}}$, where $\mathbf{v}$ is any scalar factor [1, Definition 2.1] of $\mathbf{u}$, and in general $\mathfrak{W}_{\mathbf{u}} \neq \mathfrak{W}_{\mathbf{v}}$. This provided us with many different maximal rank-1 spaces containing $\mathfrak{V}_{\mathbf{u}}$ when $\max u_i < 1$. Here we present another method for extending full left factor spaces to still other maximal rank-1 spaces, the $(\mathbf{u}, i)$-spaces, denoted $\mathfrak{W}_{\mathbf{u},i}$ (Definition 3.1) in $\mathscr{M}_{mn}(\mathbb{S})$.

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A third method for constructing maximal rank-1 spaces is introduced. This method applies to Boolean (0, 1) matrices only. Unlike the previous spaces, these maximal rank-1 spaces contain no full factor spaces (Example 5.1).

2. PRELIMINARY RESULTS

Recall the following from [1].

DEFINITION [1, Definition 2.11]. In the product

$$[\mathbf{a} \mid \mathbf{b}]\left[\frac{\mathbf{x}^t}{\mathbf{y}^t}\right]$$

(or in the sum $\mathbf{a}\mathbf{x}^t + \mathbf{b}\mathbf{y}^t$), if there exists i, j, k, and l such that $\max(a_i, y_j) < \min(a_k, b_i, y_l, x_j)$, then we say that a_i and y_j are *trapped in conjunction*. Similarly, if $\max(b_i, x_j) < \min(b_k, a_i, x_l, y_j)$, we say that b_i and x_j are *trapped in conjunction*.

It is important to note that a pair of entries which are trapped in conjunction must be from $\mathbf{a}$ and $\mathbf{y}$ or from $\mathbf{b}$ and $\mathbf{x}$ in the sum

$$\mathbf{a}\mathbf{x}^t + \mathbf{b}\mathbf{y}^t = [\mathbf{a} \mid \mathbf{b}]\left[\frac{\mathbf{x}^t}{\mathbf{y}^t}\right].$$

LEMMA [1, Lemma 2.12]. *If* $\mathbf{a}, \mathbf{b} \in \mathbb{S}^m$ *and* $\mathbf{x}, \mathbf{y} \in \mathbb{S}^n$, *then the product*

$$[\mathbf{a} \mid \mathbf{b}]\left[\frac{\mathbf{x}^t}{\mathbf{y}^t}\right]$$

is of rank 2 *if and only if it contains a pair of entires which are trapped in conjunction.*

DEFINITION [1, Definition 3.4]. Suppose $\mathbb{S}$ is a chain semiring with Dedekind completion $\mathbb{D}$. Let $\mathbf{u} \in \mathbb{D}^m$ with the property that $\mathbf{u} \neq \mathbf{0}$ and if $u_i \neq \max u_k$ then $u_i \in \mathbb{S}$. Then $\mathbf{u}$ is a *generator* over $\mathbb{S}^m$. Let $B_{\mathbf{u}} = \{b \in \mathbb{S}^m : b_i = \max b_k \text{ or } b_i \geqslant u_i\}$; then B_u is called the *left factor set generated by* $\mathbf{u}$.

DEFINITION [1, Definition 3.5]. Suppose $\mathbf{u}$ is a generator over $\mathbb{S}^m$ and $\mathbf{b} \in \mathbb{S}$ We say b_i is a *trapper subject to* $\mathbf{u}$ if $b_i > u_i$. We let $t_{\mathbf{b}}$ equal the value of the largest trapper of $\mathbf{b}$ if $\mathbf{b}$ contains a trapper, and we let $t_{\mathbf{b}} = 0$ otherwise.

Generally, "Subject to $\mathbf{u}$" is suppressed because the generator $\mathbf{u}$ is normally clear from the context.

We begin with an analog of $t_{\mathbf{b}}$.

DEFINITION 2.1. Suppose $\mathbf{u}$ is a generator over $\mathbb{S}^m$. Let $u_0 = \max u_k < 1$ and $\mathbf{b} \in \mathbb{S}^m$. For $k = 1, \ldots, m$ let

$$s(b_k) = \begin{cases} 0 & \text{if} \quad u_k = u_0, \\ 0 & \text{if} \quad u_k \leqslant u_k, \\ b_k & \text{if} \quad u_k < b_k \leqslant u_0, \\ u_0 & \text{if} \quad u_k < u_0 \leqslant b_k, \end{cases}$$

and let $s_{\mathbf{b}} = \max s(b_k)$.

It is easily seen that $s_{\mathbf{b}} \leqslant u_0$, $s_{\mathbf{b}} \leqslant t_{\mathbf{b}}$, and $s_{\alpha\mathbf{b}} \leqslant \alpha s_{\mathbf{b}}$ for all $\alpha \in \mathbb{S}$.

DEFINITION 2.2. Let $\mathbb{S}$ be a chain semiring and $\mathbf{u}$ a generator over $\mathbb{S}^m$ with maximal element $u_0 < 1$. For each $i = 1, \ldots, m$, let

$$B_{\mathbf{u},i} = \left\{\mathbf{b} \in \mathbb{S}^m : b_k \leqslant u_0 \ \forall k \neq i, \text{and}\ \forall k,\ b_k \geqslant u_k \text{ or } b_k = \max b_p\right\}.$$

Then $B_{\mathbf{u},i}$ is called the *ith factor set generated by* $\mathbf{u}$.

$B_{\mathbf{u}}$ is the set of all scalar multiples (scalars from $\mathbb{S}$) of vectors from $\mathbb{S}^m$ which absorb $\mathbf{u}$ [1, Definition 2.1]. The set $B_{\mathbf{u},i}$ has the added restriction that the ith entry of a vector in $B_{\mathbf{u},i}$ is the only entry which is allowed to be larger than $\max u_k$. Thus $B_{\mathbf{u},i} \subseteq B_{\mathbf{u}}$ for each i.

DEFINITION 2.3. Let $\mathbb{S}$ be a chain semiring and $\mathbf{u}$ a generator over $\mathbb{S}^m$ with $\max u_k < 1$. For each $i = 1, \ldots, m$ let

$$\mathfrak{T}_{\mathbf{u},i} = \left\{\mathbf{b}\mathbf{y}^t : \mathbf{b} \in B_{\mathbf{u},i}, \mathbf{y} \in \mathbb{S}^n,\ y_j \geqslant s_{\mathbf{b}}\ \forall j, \text{and} \max b_k = \max\ y_j\right\}.$$

THEOREM 2.4. *If $\mathbb{S}$ is a chain semiring and* $\mathbf{u}$ *is a generator over* $\mathbb{S}^m$ *with* $\max u_k < 1$, *then* $\mathfrak{T}_{\mathbf{u},i}$ *is a rank*-1 *space in* $\mathcal{M}_{mn}(\mathbb{S})$.

Proof. Let $\mathbf{u}$ be a generator over $\mathbb{S}^m$ with $u_0 = \max u_k < 1$.

Clearly $O \in \mathfrak{T}_{\mathbf{u},i}$.

We first prove that $\mathfrak{T}_{\mathbf{u},i}$ is closed under multiplication by scalars. For let $B \in \mathfrak{T}_{\mathbf{u},i}$ and $\alpha \in \mathbb{S}$. Then $B = \mathbf{b}\mathbf{y}^t$, where $\mathbf{b}$ and $\mathbf{y}$ have the properties given in Definition 2.3. Then $\alpha B = (\alpha\mathbf{b})(\alpha\mathbf{y})^t$. It is easy to check that $\alpha\mathbf{b} \in B_{\mathbf{u},i}$.

By Definition 2.3, for all j, $y_j \geqslant s_{\mathbf{b}}$. This implies that $\alpha y_j \geqslant \alpha s_{\mathbf{b}} \geqslant s_{\alpha\mathbf{b}}$.

Finally, $\max b_k = \max y_j$ implies $\max \alpha b_k = \max \alpha y_j$.

Therefore, $\alpha B \in \mathfrak{T}_{\mathbf{u},i}$ that is, $\mathfrak{T}_{\mathbf{u},i}$ is closed under scalar multiplication.

Next, we show that $\mathfrak{T}_{\mu,i}$ is closed under addition. We start by showing that the sum of two nonzero matrices from $\mathfrak{T}_{\mathbf{u},i}$ is rank-1.

Assume not; then there exists $\mathbf{a}$, $\mathbf{b}$, $\mathbf{x}$, and $\mathbf{y}$ such that $\mathbf{a}\mathbf{x}^t, \mathbf{b}\mathbf{y}^t \in \mathfrak{T}_{\mathbf{u}i}$ and $\mathbf{a}$, $\mathbf{b}$, $\mathbf{x}$, and $\mathbf{y}$ have the properties given in Defintiion 2.3, but $\mathbf{a}\mathbf{x}^{\mathrm{t}} + \mathbf{b}\mathbf{y}^{\mathrm{t}}$ is not rank-1. This implies, by [1, Lemma 2.12], that two entries in

$$[\mathbf{a} \mid \mathbf{b}]\left[\frac{\mathbf{x}^t}{\mathbf{y}^t}\right]$$

are trapped in conjunction. Without loss of generality, suppose a_k and y_j are trapped in conjunction. This means that there exist r and s such that $\max(a_k, y_j) < \min(a_r, b_k, y_s, x_j)$. We will consider two cases.

Case I: Suppose $u_k < u_0$. Since a_k is trapped, we know that $a_k < \max a_p$, so, by the defintion of $B_{\mathbf{u},i}$, $a_k \geqslant u_k$. But a_k trapped also implies $b_k > a_k$. Hence $b_k > u_k$. Since $u_k < u_0$, Definition 2.1 implies that $s(b_k) = \min(b_k, u_0)$. Thus, $s_{\mathbf{b}} \geqslant \min(b_k, u_0)$. If $\min(b_k, u_0) = b_k$, then $y_j \geqslant s_{\mathbf{b}} \geqslant b_k$. This contradicts the assumption that a_k and y_j are trapped in conjunction. Therefore $\min(b_k, u_0) \neq b_k$, that is, $u_0 < b_k$. By the definition of $B_{\mathbf{u},i}$, therefore, $k = i$. This implies that $\max a_p \neq a_i$. So, by the definition of $B_{\mathbf{u},i}$, $\max a_p \leqslant u_0$. We also have $u_k < u_0 < b_k$. This, in turn, implies $s(b_k) = u_0$, giving us $s_{\mathbf{b}} \geqslant u_0$. Therefore, $\max a_p \leqslant s_{\mathbf{b}} \leqslant y_j$, which contradicts the fact that a_k and y_j are trapped in conjunction.

Case II: Suppose $u_k = u_0$. In this case we show that it is impossible for a_k to be trapped. Since we are assuming a_k is trapped, we know that $a_k < \max a_p$, so, by the definition of $B_{\mathbf{u},i}$, $a_k \geqslant u_k = u_0$. Therefore, $\max a_p = a_i$, since a_i is the only entry in $\mathbf{a}$ allowed to be larger than u_0. This implies $k \neq i$, which, in turn, implies $b_k \leqslant u_0 = u_k \leqslant a_k$. Therefore a_k is not trapped.

In both cases we get a contradiction to the assumption that a_k and y_j are trapped in conjunction. So the sum of two nonzero matrices from $\mathfrak{T}_{\mathbf{u},i}$ must be rank-1 by [1, Lemma 2.12].

Now we show that $\mathfrak{T}_{\mathbf{u},i}$ is closed under addition. Let A and B be two matrices taken from $\mathfrak{T}_{\mathbf{u},i}$. If either is O, then their sum is certainly in $\mathfrak{T}_{\mathbf{u},i}$, so we will assume that A and B are nonzero. Suppose $A = \mathbf{a}\mathbf{x}^t$ and $B = \mathbf{b}\mathbf{y}^t$, where $\mathbf{a}, \mathbf{b}, \mathbf{x}$, and $\mathbf{y}$ have the properties given in Definition 2.3. Then

$$A + B = [\mathbf{a} \mid \mathbf{b}]\left[\begin{array}{c}\mathbf{x}^t \\ \hline \mathbf{y}^t\end{array}\right].$$

Let $(w_{pq}) = A + B$, suppose w_{kl} is a maximal entry in (w_{pq}). From above, we know that (w_{pq}) is rank-1, so by [1, Lemma 2.3]

$$A + B = \begin{bmatrix} w_{1l} \\ \vdots \\ w_{ml} \end{bmatrix} \begin{bmatrix} w_{k1} & \cdots & w_{kn} \end{bmatrix} = \mathbf{d}\mathbf{z}^t,$$

where

$$\mathbf{d} = \begin{bmatrix} w_{1l} \\ \vdots \\ w_{ml} \end{bmatrix} \quad \text{and} \quad \mathbf{z}^t = \begin{bmatrix} w_{k1} & \cdots & w_{kn} \end{bmatrix}.$$

So that maximal entry is $w_{kl} = a_k x_l + b_k y_l$. Without loss of generality, suppose $a_k x_l \geqslant b_k y_l$; then $w_{kl} = a_k x_l$. Since $\max a_p = \max x_q$ and w_{kl} is maximal, $w_{kl} = a_k = x_l$. But by our factorization, $w_{kl} = d_k = z_l$, so $d_k = z_l = w_{kl} = a_k = x_l$ and they are maximal entries in their respective vectors or matrices. They are also at least as large as any entry in $\mathbf{b}$ or $\mathbf{y}$,

Now we show that $\mathbf{d} \in B_{\mathbf{u},i}$.

If $1 \leqslant p \leqslant m$ then $x_l = a_k \geqslant a_p$ and

$$d_p = w_{pl} = a_p x_l + b_p y_l = a_p + b_p y_l.$$

If $a_p = a_k$, then $a_p \geqslant b_p y_l$, so $d_p = a_p = a_k = d_k = \max d_r$. And if $a_p < a_k$, then $d_p = a_p + b_p y_l \geqslant a_p \geqslant u_p$, since $a_p \neq \max a_r$. Therefore $d_p = \max d_r$ or $d_p \geqslant u_p$.

If $p \neq i$ then $a_p \leqslant u_0$ and $b_p \leqslant u_0$, by the definition of $B_{\mathbf{u},i}$, so $d_p = a_p + b_p y_l \leqslant u_0 + u_o y_l = u_0$. Therefore $\mathbf{d} \in B_{\mathbf{u},i}$.

Now we show that $\mathbf{d}\mathbf{z}^t \in \mathfrak{T}_{\mathbf{u},i}$.

Since $\mathbf{dz}^t$ is a canonical factorization of $A + B$, we have $\max d_p = \max z_q$.

If $s_{\mathbf{d}} = 0$, then clearly $z_j \geqslant s_{\mathbf{d}}$ for all j and hence $\mathbf{dz}^t \in \mathfrak{T}_{\mathbf{u},i}$.

So let $s_{\mathbf{d}} \neq 0$. If $u_k < u_0$ and the largest entry $d_k > u_k$, then $s_{\mathbf{d}} = \min(d_k, u_0)$ by definition 2.1. But $d_k = a_k$, the largest entry in $\mathbf{a}$, so $s_{\mathbf{a}} = s_{\mathbf{d}}$. Therefore, since $x_q \geqslant s_{\mathbf{a}}$ for all q, we have

$$z_q = w_{kq} = a_k x_q + b_k y_q = x_q + b_k y_q \geqslant x_q \geqslant s_{\mathbf{a}} = s_{\mathbf{d}}.$$

This implies $\mathbf{dz}^t \in \mathfrak{T}_{\mathbf{u},i}$, so we can also assume that $d_k \leqslant u_k$ or $u_k = u_0$.

Since we are assuming that $s_{\mathbf{d}} \neq 0$ and that either $d_k \leqslant u_k$ or $u_k = u_0$, by Definition 2.1 there exists a $p \neq k$ such that $u_p \neq u_0$, $d_p > u_p$, and $s_{\mathbf{d}} = \min(d_p, u_0)$. Now,

$$d_p = w_{pl} = a_p x_l + b_p y_l = a_p + b_p y_l; \tag{2.1}$$

therefore $a_p > u_p$ or $b_p y_l > u_p$.

If $1 \leqslant q \leqslant n$, we know that

$$z_q = x_q + b_k y_q. \tag{2.2}$$

All that is left to show is that $z_q \geqslant \min(d_p, u_0) = s_{\mathbf{d}}$.

Since $A, B \in \mathfrak{T}_{\mathbf{u},i}$, we have $x_q \geqslant s_{\mathbf{a}}$ and $y_q \geqslant s_{\mathbf{b}}$. If $a_p \geqslant b_p y_l$, then (2.1) gives us $d_p = a_p$, which implies $a_p > u_p$, and since $u_p \neq u_0$, we have $\min(a_p, u_0) = s(a_p)$. Therefore, from (2.2), we have

$$z_q = x_q + b_k y_q \geqslant x_q \geqslant s_{\mathbf{a}} \geqslant s(a_p) = \min(a_p, u_0) = \min(d_p, u_0) = s_{\mathbf{d}}.$$

So if $a_p \geqslant b_p y_l$, we have $\mathbf{dz}^t \in \mathfrak{T}_{\mathbf{u},i}$.

If $b_p y_l > a_p$, then (2.1) gives us $d_p = b_p y_l$, so the fact that $d_p > u_p$ implies that $b_p > u_p$. And by our choice of p, $u_p \neq u_0$, so $\min(b_p, u_0) = s(b_p) \leqslant s_{\mathbf{b}}$.

Before proving $z_q \geqslant s_{\mathbf{d}}$ when $b_p y_l > a_p$, we show that $b_k \geqslant y_q$. Consider three cases:

case I: $b_k = a_k$;
case II: $b_k < u_k$; and
case III: $u_k \leqslant b_k < a_k$.

These three cases suffice, because if neither case I nor case II holds, then $b_k \neq a_k$ and $b_k \geqslant u_k$. But we have shown that, without loss of generality, we can assume that a_k is a maximal entry in all of **a, b, d, x, y, z,** and $A + B$, so $u_k \leqslant b_k < a_k$ if I and II do not hold.

Case I: Since $b_k = a_k$, the largest of all entries, we have $b_k \geqslant y_q$.

Case II: By the definition of $B_{\mathbf{u},i}$ we have $b_k \geqslant u_k$ or $b_k = \max b_r$. In this case $b_k < u_k$, so $b_k = \max b_r$, since $\mathbf{b} \in B_{\mathbf{u},i}$. But by the definition of $\mathfrak{T}_{\mathbf{u},i}$, $\max b_r = \max y_s$, so $b_k = \max b_r = \max y_s \geqslant y_q$.

Case III: In this case we consider two subcases.

Subcase (a): *Suppose* $u_0 \leqslant b_k$. Then $u_k \leqslant u_0 \leqslant b_k < a_k$. By the definition of $B_{\mathbf{u},i}$, the only entry of $\mathbf{a}$ allowed to be larger than u_0 is the ith entry, so $k = i$. But $\mathbf{b} \in B_{\mathbf{u},i}$ also, so for all $r \neq i = k$ we have $b_r \leqslant u_0 \leqslant b_k$. Therefore $b_k = \max b_r$, and again we have $b_k = \max b_r = \max y_s \geqslant y_q$.

Subcase (b): *Suppose* $b_k < u_0$. Since $u_k \leqslant b_k$, we have $u_k \neq u_0$. Also $u_k < a_k$. Since $a_k = d_k$, we have $u_k \neq u_0$ and $u_k < d_k$. But we showed earlier that $d_k \leqslant u_k$ or $u_k = u_0$.

Therefore $b_k \geqslant y_q$.

If $b_p y_l > a_p$, then from (2.2),

$$z_q = x_q + b_k y_q = x_q + y_q \geqslant y_q \geqslant s_{\mathbf{b}} \geqslant \min\left(b_p, u_0\right)$$
$$\geqslant \min\left(b_p y_l, u_0\right) = \min\left(d_p, u_0\right) = s_{\mathbf{d}}.$$

Therefore, for all q, $z_q \geqslant s_{\mathbf{d}}$, we have $\mathbf{dz}^t \in \mathfrak{T}_{\mathbf{u},i}$. ■

Throughout the rest of this paper we shall adopt the convention that $\mathbb{S}$ *is a chain semiring with Dedekind completion* $\mathbb{D}$ *and* $\mathbf{u} \in \mathbb{D}^m$ *is a generator over* $\mathbb{S}^m$. *It is to be understood tha* u_k *is the kth entry of* $\mathbf{u}$, *a generator over* $\mathbb{S}^m$, *even if no mention is made of* $\mathbf{u}$, *or* $\mathbb{S}$.

Recall [1, Definition 3.7] the *full left factor space*

$$\mathfrak{B}_{\mathbf{u}} = \left\{(\alpha\mathbf{u})\mathbf{x}^t : \alpha \in \mathbb{S} \text{ with } \alpha \leqslant u_0, \mathbf{x} \in \mathbb{S}^n \text{ with } \max \alpha u_i = \max x_j\right\}.$$

LEMMA 2.5. *If* $\max u_k < 1$, *then* $\mathfrak{B}_{\mathbf{u}} \subseteq \mathfrak{T}_{\mathbf{u},i}$ *for all* i.

Proof. Let $u_0 = \max u_k < 1$ and $A \in \mathfrak{B}_{\mathbf{u}}$, then $A = \mathbf{ax}^t$ where $\mathbf{a} = \alpha\mathbf{u}$ for some $\alpha \in \mathbb{S}$ and $\mathbf{ax}^t$ is the canonical factorization of A. Since $\mathbf{a} = \alpha\mathbf{u}$, we know $a_k \leqslant u_0$ for all k, and either $a_k = u_k$ or else $a_k = \max a_p$. Therefore $\mathbf{a} \in B_{\mathbf{u},i}$.

Since $a_k \leqslant u_k$ for all k, $s_{\mathbf{a}} = 0$ and so $x_j \geqslant s_{\mathbf{a}}$ for all j.

Since $\mathbf{ax}^t$ is canonical, $\max a_k = \max x_j$.

Therefore $A \in \mathfrak{T}_{\mathbf{u},i}$ for all i. ■

DEFINITION 2.6. If $\max u_k < 1$, then for each $i = 1, \ldots, m$ let

$$C_{\mathbf{u},i} = \left\{\mathbf{c} \in \mathbb{S}^m : c_i = \max c_p \text{ and } \forall k\, c_k \geqslant u_k \text{ or } c_k = \max c_p\right\},$$

and let

$$\mathfrak{U}_{\mathbf{u},i} = \left\{\mathbf{c}\mathbf{z}^t : \mathbf{c} \in C_{\mathbf{u},i}, \mathbf{z} \in \mathbb{S}^n, \max c_k = \max z_j, \text{ and } \forall j\, z_j \geqslant t_c\right\}.$$

The set $C_{\mathbf{u},i}$ is the set of all scalar multiples of vectors which absorb $\mathbf{u}$ and have their largest entry in the ith position.

Recall [1, Definition 3.6]

$$\mathfrak{W}_{\mathbf{u}} = \left\{\mathbf{b}\mathbf{y}^t : \mathbf{b} \in B_{\mathbf{u}}, \mathbf{y} \in \mathbb{S}^n, \max b_i = \max y_j, \text{ and } \forall_j\, y_j \geqslant t_{\mathbf{b}}\right\}.$$

LEMMA 2.7. *If* $\max u_k < 1$, *then* $\mathfrak{U}_{\mathbf{u},i}$ *is a rank*-1 *space.*

Proof. It is clear that $O \in \mathfrak{U}_{\mathbf{u},i}$.

Let $A, B \in \mathfrak{U}_{\mathbf{u},i}$, where $A = \mathbf{a}\mathbf{x}^t$, $B = \mathbf{b}\mathbf{y}^t$, and $\mathbf{a}, \mathbf{b}, \mathbf{x}$, and $\mathbf{y}$ satisfy the requirements of Definition 2.6.

Let $\alpha \in \mathbb{S}$. Since $\mathfrak{U}_{\mathbf{u},i} \subseteq \mathfrak{W}_{\mathbf{u}}$, $\alpha A \in \mathfrak{W}_{\mathbf{u}}$. But $a_i \geqslant a_p$ for all p implies $\alpha a_i \geqslant \alpha a_p$ for all p, so $\alpha a \in C_{\mathbf{u},i}$. Since this is the only new restriction imposed by the definition of $\mathfrak{U}_{\mathbf{u},i}$ over the definition of $\mathfrak{W}_{\mathbf{u}}$, we have $\alpha A \in \mathfrak{U}_{\mathbf{u},i}$. Therefore $\mathfrak{U}_{\mathbf{u},i}$ is closed under scalar multiplication.

Since $\mathfrak{U}_{\mathbf{u},i} \subseteq \mathfrak{W}_{\mathbf{u}}$, $A + B$ is O or has rank 1. So there exist $\mathbf{d} \in \mathbb{S}^m$ and $\mathbf{z} \in \mathbb{S}^n$ such that $\mathbf{d}\mathbf{z}^t$ is the canonical factorization of $A + B$. Since $\mathfrak{W}_{\mathbf{u}}$ is a rank-1 space, we know that $A + B \in \mathfrak{W}_{\mathbf{u}}$. There is only one extra condition which must be satisfied to show that $A + B \in \mathfrak{U}_{\mathbf{u},i}$. That condition is $d_i = \max d_p$.

We are given that $\mathbf{a}, \mathbf{b} \in C_{\mathbf{u},i}$, so $a_i = \max a_p$ and $b_i = \max b_p$. But for some l, $d_i = a_i x_l + b_i y_l$; thus $d_i = a_i x_l + b_i y_l \geqslant a_p x_l + b_p y_l = d_p$. So $A + B \in \mathfrak{U}_{\mathbf{u},i}$, and $\mathfrak{U}_{\mathbf{u},i}$ is a rank-1 space. ■

LEMMA 2.8. *Let* $\max u_k < 1$. *If B and C are nonzero, with* $B \in \mathfrak{T}_{\mathbf{u},i}$ *and* $C \in \mathfrak{U}_{\mathbf{u},i}$, *then* $B + C$ *is rank*-1.

Proof. Suppose $\mathbf{u}$ is a generator over $\mathbb{S}^m$ with $u_0 = \max u_k < 1$. Let $B = \mathbf{b}\mathbf{y}^t$ and $C = \mathbf{c}\mathbf{z}^t$, where $\mathbf{b}$ and $\mathbf{y}$ satisfy Definition 2.3 and $\mathbf{c}$ and $\mathbf{z}$ satisfy Definition 2.6. Then

$$B + C = [\mathbf{b} \mid \mathbf{c}]\left[\begin{array}{c}\mathbf{y}^t \\ \hline \mathbf{z}^t\end{array}\right].$$

If no entries of $[\mathbf{b} \mid \mathbf{c}]$ are trapped, then $B + C$ is rank-1 by [1, Lemma 2.12]. So we can assume that some entry in $[\mathbf{b} \mid \mathbf{c}]$ is trapped.

Suppose b_k is trapped by c_k and b_p. Then $b_k \neq \max b_r$, and by the definition of $B_{\mathbf{u},i}$, $b_k \geqslant u_k$. This implies $c_k > u_k$, so c_k is a trapper. Therefore $t_{\mathbf{c}} \geqslant c_k$. But by the definition of $\mathfrak{U}_{\mathbf{u},i}$, $z_j \geqslant t_{\mathbf{c}}$ for all j, so no entry in $\mathbf{z}$ can be trapped in conjunction with b_k.

Suppose c_k is trapped. Then since $c_i = \max c_r$, we know $k \neq i$ and c_k is trapped by c_i and b_k. As above, $c_k \neq \max c_r$ implies $c_k \geqslant u_k$, so $b_k > u_k$, since b_k traps c_k. Now, $k \neq i$ implies $b_k \leqslant u_0$ by the definition of $B_{\mathbf{u},i}$, so $u_k < b_k \leqslant u_0$, which implies $s_{\mathbf{b}} \geqslant s(b_k) = b_k$. But by the definition of $\mathfrak{T}_{\mathbf{u},i}$, $y_j \geqslant s_{\mathbf{b}}$ for all j, so $y_j \geqslant b_k$. Therefore no entry in $\mathbf{y}$ can be trapped in conjunction with c_k.

Since no entry in $\left[\begin{array}{c}\mathbf{y}^t\\ \hline \mathbf{z}^t\end{array}\right]$ can be trapped in conjunction with an entry in $[\mathbf{b} \mid \mathbf{c}]$, $B + C$ is rank-1 by [1, Lemma 2.12]. ■

LEMMA 2.9. *The set* $\mathfrak{T}_{\mathbf{u},i} + \mathfrak{U}_{\mathbf{u},i}$ *is a rank*-1 *space.*

Proof. Since $O \in \mathfrak{T}_{\mathbf{u},i} \cap \mathfrak{U}_{\mathbf{u},i}$, we have $O \in \mathfrak{T}_{\mathbf{u},i} + \mathfrak{U}_{\mathbf{u},i}$.

We must show that all elements of $\mathfrak{T}_{\mathbf{u},i} + \mathfrak{U}_{\mathbf{u},i}$ are either O or rank-1, and we must show that $\mathfrak{T}_{\mathbf{u},i} + \mathfrak{U}_{\mathbf{u},i}$ is closed under scalar multiplication and under addition.

An arbitrary element of $\mathfrak{T}_{\mathbf{u},i} + \mathfrak{U}_{\mathbf{u},i}$ looks like $B + C$ where $B \in \mathfrak{T}_{\mathbf{u},i}$ and $C \in \mathfrak{U}_{\mathbf{u},i}$. If B or C is O, then $B + C$ is clearly O or rank-1, since $\mathfrak{T}_{\mathbf{u},i}$ and $\mathfrak{U}_{\mathbf{u},i}$ are rank-1 spaces. If neither B nor C is O, Lemma 2.8 implies $B + C$ is rank-1.

Since $\mathfrak{T}_{\mathbf{u},i}$ and $\mathfrak{U}_{\mathbf{u},i}$ are closed under addition and multiplication by scalars, so is $\mathfrak{T}_{\mathbf{u},i} + \mathfrak{U}_{\mathbf{u},i}$.

Therefore $\mathfrak{T}_{\mathbf{u},i} + \mathfrak{U}_{\mathbf{u},i}$ is a rank-1 space. ■

3. THE MAIN THEOREM

DEFINITION 3.1. Let $\mathfrak{W}_{\mathbf{u},i} = \mathfrak{T}_{\mathbf{u},i} + \mathfrak{U}_{\mathbf{u},i}$. The space $\mathfrak{W}_{\mathbf{u},i}$ is called a $(\mathbf{u}, i)$ *space* in $\mathscr{M}_{mn}(\mathbb{S})$.

DEFINITION 3.2. Let $u_0 = \max u_k$. If $\mathbf{u} \notin \mathbb{S}^m$, and there exists a $p \neq i$ such that $u_p = u_0$, then i is said to be *exceptional* (subject to $\mathbf{u}$).

So i is not exceptional if and only if either u_i is the unique largest entry in $\mathbf{u}$ or $u_0 \in \mathbb{S}$.

THEOREM 3.3. *Let $\mathbb{S}$ be a chain semiring,* $\mathbf{u}$ *a generator over* $\mathbb{S}^m$ *with* $\max u_k < 1$, *and suppose i is not exceptional. Then* $\mathfrak{W}_{\mathbf{u},i}$ *is a maximal rank-1 space.*

Proof. By Lemma 2.9, we need only show maximality. For ease of notation, suppose $u_1 \geqslant \cdots \geqslant u_m$. If $\alpha \in \mathbb{S}$ with $\alpha < u_1$, there exists $u_1 - \varepsilon \in \mathbb{S}$ with $\alpha < u_1 - \varepsilon \leqslant u_1$, for if $u_1 \in \mathbb{S}$, then $u_1 - \varepsilon = u_1$ works, and if $u_1 \notin \mathbb{S}$, the result follows from the fact that $u_1 \in \mathbb{D}$, the Dedekind completion of $\mathbb{S}$. Likewise, if $u_1 < \alpha$ then there exists $u_1 + \varepsilon \in \mathbb{S}$ such that $u_1 \leqslant u_1 + \varepsilon < \alpha$.

Suppose that for all $A \in \mathfrak{T}_{\mathbf{u},i} + \mathfrak{U}_{\mathbf{u},i}$, the matrix $A + D$ is O or rank-1. We now show that $D \in \mathfrak{T}_{\mathbf{u},i} + \mathfrak{U}_{\mathbf{u},i}$.

Since $O \in \mathfrak{T}_{\mathbf{u},i} + \mathfrak{U}_{\mathbf{u},i}$, we can assume $D \neq O$. It must be true that D is rank-1, for otherwise $O + D$ is not rank-1. Let $D = \mathbf{d}\mathbf{v}^t$ be the canonical factorization of D.

If $\mathbf{d} \in B_{\mathbf{u},i}$, then $\mathbf{d}$ satisfies the conditions to place $\mathbf{d}\mathbf{v}^t$ in $\mathfrak{T}_{\mathbf{u},i} \subseteq \mathfrak{T}_{\mathbf{u},i} + \mathfrak{U}_{\mathbf{u},i}$, so we can assume that $\mathbf{v}$ does not satisfy the conditions of the definition of $\mathfrak{T}_{\mathbf{u},i}$. Since $\mathbf{d}\mathbf{v}^t$ is canonical, $\max d_p = \max v_q$, so we assume that there exists j such that $v_j < s_{\mathbf{d}}$. This implies that $s_{\mathbf{d}} \neq$ O, so there exists k such that $u_k < \min(d_k, u_1) = s_{\mathbf{d}} \leqslant u_1$. This makes both u_k and v_j strictly smaller than u_1, so there exists $u_1 - \varepsilon \in \mathbb{S}$ such that $\max(u_k, v_j) < u_1 - \varepsilon \leqslant u_1$. Let $\mathbf{b} = (u_1 - \varepsilon)\mathbf{u}$ and $\mathbf{y}^t = [u_1 - \varepsilon, \dots, u_1 - \varepsilon]$. Then $\mathbf{b}\mathbf{y}^t \in \mathfrak{T}_{\mathbf{u},i}$, but b_k $(= u_k)$ and v_j are trapped in conjuction in the sum $\mathbf{b}\mathbf{y}^t + \mathbf{d}\mathbf{v}^t$. This is a contradiction, so $\mathbf{d}$ and $\mathbf{v}$ satisfy the conditions of $\mathfrak{T}_{\mathbf{u},i}$. In this case, therefore, $D \in \mathfrak{T}_{\mathbf{u},i} \subseteq \mathfrak{T}_{\mathbf{u},i} + \mathfrak{U}_{\mathbf{u},i}$

Suppose $\mathbf{d} \notin B_{\mathbf{u},i}$. Either there exists $k \neq i$ such that $d_k > u_1$ or else there exists k such that $d_k < u_k$ and $d_k \neq \max d_p$.

Suppose there exists k such that $d_k < u_k$ and $d_k \neq \max d_p$. By assumption $u_k \leqslant u_1$, so there exists $u_1 - \varepsilon \in \mathbb{S}$ such that $d_k < u_1 - \varepsilon \leqslant u_1$. Since $\mathbf{u}$ is a generator over $\mathbb{S}^m$ [1, Definition 3.4], u_1-$\varepsilon \in \mathbb{S}$, and $0 < u_1 - \varepsilon \leqslant u_1$, we have $(u_1 - \varepsilon)\mathbf{u} \in B_{\mathbf{u}}$. Since $\mathbf{d}\mathbf{v}^t$ is canonical, there exists j such that $v_j = \max d_p$. Define $\mathbf{a} \in \mathbb{S}^m$ and $\mathbf{x} \in \mathbb{S}^n$ as follows: Let $\mathbf{a} = (u_1 - \varepsilon)\mathbf{u}$, let $x_q = u_1 - \varepsilon$ for $q \neq j$, and let $x_j = d_k$. Then $\mathbf{a}\mathbf{x}^t \in \mathsf{V}_{\mathbf{u}} \subseteq \mathfrak{T}_{\mathbf{u},i}$ (Lemma 2.5). But d_k and x_j are trapped in conjunction in the sum $\mathbf{a}\mathbf{x}^t + \mathbf{d}\mathbf{v}^t$. This contradicts our assumption that the sum is rank-1. Therefore, we can assume that there exists $k \neq i$ such that $d_k > u_1$. We choose k so that $d_k = \max_{p \neq i} d_p$.

Suppose $\mathbf{d} \notin C_{\mathbf{u},i}$. We showed above that assuming there exists a p such that $d_p < u_p$ and $d_p \neq \max d_r$ leads to a contradiction. Therefore $d_i < \max d_r = \max_{r \neq i} d_r = d_k$. Define $\mathbf{b} \in \mathbb{S}^m$ as follows: Let $b_i = d_k$, and let $b_p = u_p$ for $p \neq i$. Since i is not exceptional, $\mathbf{b} \in B_{\mathbf{u},i}$. Since $u_1 < d_k$, there exists $u_1 + \varepsilon \in \mathbb{S}$ such that $u_1 \leqslant u_1 + \varepsilon < d_k$. Since the factorization $\mathbf{d}\mathbf{v}^t$ is canonical, there exists j such that $v_j = d_k$. Define $\mathbf{y} \in \mathbb{S}^n$ as follows: Let

$y_j = u_1 + \varepsilon$, and let $y_q = d_k$ for $q \neq j$. Since $s(b_p) = 0$ for $p \neq i$, we have $s_{\mathbf{b}} = s(b_i) = u_1$. So $y_q \geqslant s_{\mathbf{b}}$ for all q, and since $\max b_p = d_k = \max y_q$, we have $\mathbf{b}\mathbf{y}^t \in \mathfrak{T}_{\mathbf{u},i}$. But d_i and y_j are trapped in conjunction in the sum $\mathbf{b}\mathbf{y}^t + \mathbf{d}\mathbf{v}^t$. This contradicts the fact that $A + D$ is rank-1 for all $A \in \mathfrak{T}_{\mathbf{u},i} + \mathfrak{U}_{\mathbf{u},i}$. Therefore, $\mathbf{d} \in C_{\mathbf{u},i}$.

Because we are able to assume $\mathbf{d} \in C_{\mathbf{u},i}$, we have $d_i = \max d_p$. We are still able to assume that there exists $k \neq i$ such that $d_k > u_1$, and then we choose k such that $d_k = \max_{p \neq i} d_p$. Putting these together, we have $u_1 < d_k \leqslant d_i$.

If $\mathbf{v}$ satisfies the conditions of Definition 2.6, then $\mathbf{d}\mathbf{v}^t \in \mathfrak{U}_{\mathbf{u},i} \subseteq \mathfrak{T}_{\mathbf{u},i} + \mathfrak{U}_{\mathbf{u},i}$ and we are done. So we can assume $\mathbf{v}$ does not satisfy those conditions. Since $\mathbf{d}\mathbf{v}^t$ is canonical, $\max d_p = \max v_q$, and thus we can assume that there exists j such that $v_j < t_{\mathbf{d}}$.

But what is $t_{\mathbf{d}}$? Since $u_i \leqslant u_1 < d_i$, we know that d_i is a trapper. Now, d_i a trapper and $d_i = \max d_p$ imply $t_{\mathbf{d}} = d_i$.

We know that there exists j such that $v_j < d_i$, but we don't know which side of d_k that v_j falls on.

Suppose there exists j such that $v_j < d_k$. Then define $\mathbf{c} \in \mathbb{S}^m$ and $\mathbf{z} \in \mathbb{S}^n$ as follows: Let $c_k = u_1 + \varepsilon$ and $c_p = d_k$ for $p \neq k$. Let $z_q = d_k$ for all q. Then by Definition 2.6, $\mathbf{c}\mathbf{z}^t \in \mathfrak{U}_{\mathbf{u},i}$. But c_k and v_j are trapped in conjunction in the sum $\mathbf{c}\mathbf{z}^t + \mathbf{d}\mathbf{v}^t$. This is a contradiction. So for all j, $v_j \geqslant d_k$.

We have now restricted $\mathbf{d}$ and $\mathbf{v}$ enough to be able to show $\mathbf{d}\mathbf{v}^t \in \mathfrak{T}_{\mathbf{u},i} + \mathfrak{U}_{\mathbf{u},i}$.

Define $\mathbf{b}, \mathbf{c} \in \mathbb{S}^m$ and $\mathbf{y}, \mathbf{z} \in \mathbb{S}^n$ as follows.

Let $b_i = d_i$, and for $p \neq i$, let $b_p = u_p$. Since i is not exceptional, $\mathbf{b} \in \mathbb{S}^m$. In fact, $\mathbf{b} \in B_{\mathbf{u},i}$. Since $b_p = u_p$ for $p \neq i$, we have $s(b_p) = 0$ for $p \neq i$ and therefore $s_{\mathbf{b}} = s(b_i) = u_1$.

Let $\mathbf{y} = \mathbf{v}$. Since $v_j \geqslant d_k$ for all j, we know that $v_j \geqslant d_k > u_1 = s_{\mathbf{b}}$ and

$$\max b_p = b_i = d_i = \max d_p = \max v_q = \max y_q.$$

So $\mathbf{b}\mathbf{y}^t$ is canonical. Therefore $\mathbf{b}\mathbf{y}^t \in \mathfrak{T}_{\mathbf{u},i}$

Let $\mathbf{c} = d_k\mathbf{d}$. Clearly, $d_i = \max d_p$ implies $c_i = \max c_p$, and $d_p \geqslant u_p$ or $d_p = \max d_r$ implies $c_p \geqslant u_p$ or $c_p = \max c_r$. Therefore $c \in C_{\mathbf{u},i}$.

Let $z_q = d_k$ for all q. Then $\max c_p = d_k = z_q$, and for all q, $z_q = d_k = t_{\mathbf{c}}$. Therefore, $\mathbf{c}\mathbf{z}^t \in \mathfrak{U}_{\mathbf{u},i}$.

Now we show that $\mathbf{d}\mathbf{v}^t = \mathbf{b}\mathbf{y}^t + \mathbf{c}\mathbf{z}^t$.

Let $(w_{pq}) = \mathbf{d}\mathbf{v}^t$. First, $w_{iq} = d_i v_q = v_q$, since $d_i = \max d_r$ and $\mathbf{d}\mathbf{v}^t$ is canonical. But

$$b_i y_q + c_i z_q = d_i v_q + (d_k d_i) d_k = v_q + d_k = v_q.$$

since $v_q \geqslant d_k$. So, the ith rows are the same.

For $p \neq i$, $w_{pq} = d_p v_q = d_p$, since $v_q \geqslant d_k = \max_{r \neq i} d_r$. On the other hand,

$$b_p y_q + c_p z_q = u_p v_q + (d_k d_p) d_k = u_p + d_p = d_p,$$

since $\mathbf{d} \in C_{\mathbf{u},i}$ guarantees $d_p \geqslant u_p$ or $d_p = \max d_r$, and $\min d_r = d_i > u_1 \geqslant u_p$. Therefore, the other rows match too.

Summarizing, either $D \in \mathfrak{T}_{\mathbf{u},i} + \mathfrak{U}_{\mathbf{u},i}$ or else $\mathfrak{T}_{\mathbf{u},i} + \mathfrak{U}_{\mathbf{u},i}$ contains a matrix which when added to D results in a matrix of rank greater than one. But by our choice of D, for all $A \in \mathfrak{T}_{\mathbf{u},i} + \mathfrak{U}_{\mathbf{u},i}$, $A + D$ is O or of rank-1. Therefore, $D \in \mathfrak{T}_{\mathbf{u},i} + \mathfrak{U}_{\mathbf{u},i}$. This proves that $\mathfrak{T}_{\mathbf{u},i} + \mathfrak{U}_{\mathbf{u},i}$ is a maximal rank-1 space. ∎

4. EXAMPLES

It was clearly pointed out where the technical condition "i is not exceptional" was used in the above proof. What isn't so clear is what $\mathfrak{T}_{\mathbf{u},i} + \mathfrak{U}_{\mathbf{u},i}$ looks like when i is exceptional.

Suppose i is exceptional and $\mathbf{b}\mathbf{y}^t \in \mathfrak{T}_{\mathbf{u},i}$; then there exists $k \neq i$ such that $u_k = \max u_p = u_0$ and $u_0 \notin \mathbb{S}$. So, by the definition of $B_{\mathbf{u},i}$, we have $b_k < u_k$. Also by the definition of $B_{\mathbf{u},i}$, we have $b_k \geqslant u_k$ or $b_k = \max b_p$. Since we can't have $b_k < u_k$ and $b_k \geqslant u_k$ at the same time, we must have $\max b_p = b_k < u_k = u_0$. This implies $b_p < u_0$ for all p. It is clear from the definition that $B_{\mathbf{u},i}$ is a proper subset of $B_{\mathbf{u}}$. In addition, $b_p < u_0$ for all p, and thus $s_{\mathbf{b}} = t_{\mathbf{b}}$. So the conditions $\mathfrak{T}_{\mathbf{u},i}$ places on $\mathbf{y}$ are identical to those of $\mathfrak{W}_{\mathbf{u}}$. Therefore, $\mathfrak{T}_{\mathbf{u},i} \subseteq \mathfrak{W}_{\mathbf{u}}$. We already know that $\mathfrak{U}_{\mathbf{u},i} \subseteq \mathfrak{W}_{\mathbf{u}}$, so $\mathfrak{T}_{\mathbf{u},i} + \mathfrak{U}_{\mathbf{u},i} \subseteq \mathfrak{W}_{\mathbf{u}}$. The containment can be proper, as the following example shows.

Example 4.1. Let $\mathbb{S}$ be the rationals between 0 and 1, and let

$$\mathbf{u} = \begin{bmatrix} 1/e \\ 0.2 \end{bmatrix}.$$

Consider the rank-1 subspace $\mathfrak{T}_{\mathbf{u},2} + \mathfrak{U}_{\mathbf{u},2}$ of $\mathscr{M}_{22}(\mathbb{S})$. The set $\mathbb{D} = [0, 1]$ is the Dedekind completion of $\mathbb{S}$. The integer 2 is exceptional in this case, because $u_1 = 1/e \notin \mathbb{S}$. We have shown above that $\mathfrak{T}_{\mathbf{u},2} + \mathfrak{U}_{\mathbf{u},2} \subseteq \mathfrak{W}_{\mathbf{u}}$. To see that the containment is proper, look at

$$D = \begin{bmatrix} 1 \\ 0.2 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0.2 & 2 \end{bmatrix}.$$

Clearly $D \in \mathfrak{W}_{\mathbf{u}}$, but $D \notin \mathfrak{T}_{\mathbf{u},i} + \mathfrak{U}_{\mathbf{u},i}$. To see this, assume that $D \in \mathfrak{T}_{\mathbf{u},i} + \mathfrak{U}_{\mathbf{u},i}$; then $D = \mathbf{b}\mathbf{y}^t + \mathbf{c}\mathbf{z}^t$, where $\mathbf{b}\mathbf{y}^t \in \mathfrak{T}_{\mathbf{u},2}$ and $\mathbf{c}\mathbf{z}^t \in \mathfrak{U}_{\mathbf{u},2}$. We have

$1 = d_{11} = b_1 y_1 + c_1 z_1$, so either $b_1 = y_1 = 1$ or $c_1 = z_1 = 1$. But $b_1 \leqslant 1/e$, so $c_1 = z_1 = 1$. The restriction $c_2 = \max c_p$ implies $c_2 = 1$ also; therefore $0.2 = d_{21} = b_2 y_1 + c_2 z_1 = c_2 z_1 = (1)(1) = 1$. This is a contradiction, so $D \notin \mathfrak{T}_{\mathbf{u},2} + \mathfrak{U}_{\mathbf{u},2}$. Thus $\mathfrak{T}_{\mathbf{u},2} + \mathfrak{U}_{\mathbf{u},2}$ is not a maximal rank-1 space.

We now demonstrate, by example, that the class of maximal rank-1 spaces described here [the $(\mathbf{u}, i)$-spaces] is not contained in the class of $\mathbf{u}$-spaces described in [1].

EXAMPLE 4.2. Let

$$\mathbb{S} = [0,1] \quad \text{and} \quad \mathbf{u} = \begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix}.$$

Note that 2 is not exceptional, so $\mathfrak{T}_{\mathbf{u},2} + \mathfrak{U}_{\mathbf{u},2}$ in $\mathscr{M}_{22}(\mathbb{S})$ is a maximal rank-1 space.

The matrices

$$\begin{bmatrix} 0.7 & 0.7 \\ 0.3 & 0.3 \end{bmatrix} = \begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix} \begin{bmatrix} 0.7 & 0.7 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0.7 & 0.7 \\ 1 & 0.7 \end{bmatrix} = \begin{bmatrix} 0.7 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0.7 \end{bmatrix}$$

are contained in $\mathfrak{T}_{\mathbf{u},2} \subseteq \mathfrak{T}_{\mathbf{u},2} + \mathfrak{U}_{\mathbf{u},2}$. These two matrices are not contained in $\mathfrak{W}_{\mathbf{v}}$ for any $\mathbf{v}$ where $\max v_k = 1$, because different rows contain the maximal entries of the matrices. The second matrix is not contained in $\mathfrak{W}_{\mathbf{w}}$ for any $\mathbf{w}$ where $\max w_k < 1$, because in $\mathfrak{W}_{\mathbf{w}}$, if a matrix contains a 1, then the whole row where it occurs must be 1's, since 1 is a trapper.

Therefore the class of $(\mathbf{u}, i)$ = spaces is not contained in the class of $\mathbf{u}$-spaces.

Example 4.3 shows that the class of $\mathbf{u}$-spaces is not contained in the class of $(\mathbf{u}, i)$-spaces.

EXAMPLE 4.3. Let

$$\mathbb{S} = [0,1] \quad \text{and} \quad \mathbf{u} = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}.$$

Let $\mathfrak{W}_{\mathbf{u}}$ be the $\mathbf{u}$-space of $\mathscr{M}_{22}(\mathbb{S})$ described in [1]. The matrices

$$\begin{bmatrix} 1 \\ 0.5 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0.5 & 0.5 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.5 \\ 1 & 1 \end{bmatrix}$$

are in $\mathfrak{W}_{\mathbf{u}}$, but they are not both contained in any $(\mathbf{u}, i)$-space.

In order to have

$$\begin{bmatrix} 1 & 1 \\ 0.5 & 0.5 \end{bmatrix} \in \mathfrak{T}_{\mathbf{v},i} + \mathfrak{U}_{\mathbf{v},i},$$

we must have $i = 1$. But in order to have

$$\begin{bmatrix} 0.5 & 0.5 \\ 1 & 1 \end{bmatrix} \in \mathfrak{T}_{\mathbf{v},i} + \mathfrak{U}_{\mathbf{v},i}$$

we must have $i = 2$. This is because if

$$\begin{bmatrix} 1 & 1 \\ 0.5 & 0.5 \end{bmatrix} = \mathbf{b}\mathbf{y}^t + \mathbf{c}\mathbf{z}^t,$$

either $b_1 = y_1 = 1$ or $c_1 = z_1 = 1$. If $b_1 = y_1 = 1$,, then $i = 1$, since the ith entry of $\mathbf{b}$ is the only entry allowed to be as large as 1. If $c_1 = z_1 = 1$, then since $c_i = \max c_p$, we have $i = 1$ or $c_2 = 1$. But $c_2 = 1$ implies that $0.5 = b_2 y_1 + c_2 z_1 \geqslant c_2 z_1 = (1)(1) = 1$. Therefore $i = 1$.

A similar argument shows that if

$$\begin{bmatrix} 0.5 & 0.5 \\ 1 & 1 \end{bmatrix} \in \mathfrak{T}_{\mathbf{v},i} + \mathfrak{U}_{\mathbf{v},i},$$

then $i = 2$.

Since i cannot be both 1 and 2 at the same time, $\mathfrak{W}_{\mathbf{u}}$ is not a $(\mathbf{u}, i)$-space.

5. COUNTEREXAMPLE

All of the maximal rank-1 spaces which we have constructed so far have been obtained by extending a full factor space. We can demonstrate that not all maximal rank-1 spaces can be constructed this way by exhibiting a maximal rank-1 space which contains no full factor space.

EXAMPLE 5.1. *Let $\mathbb{B}$ be the Boolean $(0, 1)$ semiring. For $i, j \leqslant n$ let $A_{ij} \in \mathscr{M}_{nn}(\mathbb{B})$ such that $a_{pq} = 1$ if $p \leqslant i$ and $q \leqslant j$ and $a_{pq} = 0$ otherwise. Similarly, let $E_{ij} \in \mathscr{M}_{nn}(\mathbb{B})$ such that $e_{pq} = 1$ if $p = i$ and $q = j$ and $e_{pq} = 0$ otherwise. For $n \geqslant 2$, let $\mathfrak{A}_n = \{A_{ij} : j = i \text{ or } j = i + 1\} \cup \{E_{12}, O\}$. We will show that $\mathfrak{A}_n$ is a maximal rank-1 space and that for $n > 2$, $\mathfrak{A}_n$ contains no nonzero factor spaces.*

Order the nonzero non-E_{12} elements of $\mathfrak{A}_n$ as follows: $A_{11} < A_{12} < A_{22} < A_{23} < \cdots < A_{i,i} < A_{i,i+1} < \cdots < A_{n-1,n} < A_{n,n}$. Notice that if $A_{ij} \leqslant A_{kl}$, then $A_{ij} + A_{kl} = A_{kl}$. The sum $A_{11} + E_{12} = A_{12}$, and for all $A_{ij} \in \mathfrak{A}_n$

except A_{11} we have $A_{ij} + E_{12} = A_{ij}$, and so $\mathfrak{A}_n$ is closed. Clearly, all nonzero elements of $\mathfrak{A}_n$ have a Boolean rank of one, so $\mathfrak{A}_n$ is a Boolean rank-1 space.

To show that $\mathfrak{A}_n$ is maximal, suppose B is a rank-1 matrix which is not in $\mathfrak{A}_n$. We will show that there exists an element in $\mathfrak{A}_n$ which when added to B will result in a matrix which is not rank-1.

If for all the elements A_{ij} which have been ordered above, $B + A_{ij} = B$, then $B = A_{nn}$, which is impossible, since $B \notin \mathfrak{A}_n$. So let A_{ij} be the first in the above ordering such that $B + A_{ij} \neq B$. If $i = j = 1$, then $b_{11} = 0$, so by [1, Lemma 2.9], either the first row of B is all 0's or the first column is all 0's.

Suppose the first row of B is all 0's. Then either $B + A_{11}$ is not rank-1 or every column of B but the first is made up of 0's, by [1, Lemma 2.9]. Now, B is a nonzero matrix, so it contains at least one 1 and, by the above, the first column is the only place it can be. We know that the (1, 1) entry of B is 0, so suppose $b_{k1} = 1$ where $k > 1$. Then $B + A_{12}$ is not rank-1, since the $(k, 2)$ entry in $B + A_{12}$ is 0 and is trapped by the 1's in positions $(k, 1)$ and $(1, 2)$ [1, Definition 2.10].

Suppose the first column of B is all 0's. Then either $B + A_{11}$ is not rank-1 or every row of B but the first is made up of 0's, by [1, Lemma 2.9]. As above, the first row of B must contain a 1 outside of position (1, 1). If it contains only one in the (1, 2) position, then $B = E_{12} \in \mathfrak{A}_n$, so we can assume $b_{1l} = 1$, where $l \neq 1, 2$. Then $B + A_{22}$ is not rank-1, because the $(2, l)$ entry in the sum is 0 and is trapped by the 1's in positions (2, 1) and $(1, l)$.

If $j = i$ with $i > 1$, then there exists a q with $1 \leqslant q \leqslant i$ such that $b_{iq} = 0$. Since B is rank-1 and $B + A_{i-1,i} = B$, [1, Lemma 2.9] implies the ith row of B is made of 0's; thus $i < n$, for otherwise $B = A_{n-1,n}$. Now, either $B + A_{ii}$ is not rank-1 or else columns $i + 1$ through n are 0's. Since $B \notin \mathfrak{A}_n$, there exist s, t such that $b_{st} = 1$, $i < s \leqslant n$, and $1 \leqslant t \leqslant i$. But now we know that $B + A_{i,i+1}$ is not rank-1, because the 0 in the entry $(s, i + 1)$ is trapped by the 1's in positions (s, t) and $(1, i + 1)$. A similar argument will hold if $j = i + 1$ by switching the roles of rows and columns. Therefore $\mathfrak{A}_n$ is a maximal rank-1 space.

Suppose $n > 2$ and $\mathbf{u}$ is a nonzero vector in $\mathbb{B}^n$. Then clearly $\mathbf{u}\mathbf{e}_n^t \in \mathfrak{V}_{\mathbf{u}} \subseteq \mathfrak{W}_{\mathbf{u}}$. But $\mathbf{u}\mathbf{e}_n^t \notin \mathfrak{A}_n$, since it has nonzero entries in the nth column only. A similar argument shows that $\mathfrak{A}_n$ contains no right factor spaces.

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